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Invariant behaviour classes for the response of simple fractal circuits

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Abstract. The response function of some simple fractal circuit systems are discussed. In addition to the ubiquitous constant phase angle response (CPA), a second behaviour class is identified and its dynamic scaling equation determined. The conditions under which each of the invariant behaviour classes occur have been derived. Computed response functions are used to illustrate the formal theoretical results. An outline is also given of the relationship of the circuit systems to experimental dielectric responses for which both behaviour classes can be observed.

1. Introduction

Attention has been paid, since the work of Liu (1985) to the development of fractional power-law dispersive behaviour in the impedance or admittance of electrical circuits which contain self-similar scaling. Liu (1985) showed that for some specific circuits of this type the electrical impedance (Z) at radian frequency ω could be expressed (over a limited frequency range) in terms of the equivalent impedance at the scaled frequency ($\xi\omega$) in the form

$$Z(\omega) = \kappa Z(\xi\omega). \tag{1}$$

This equation defines the existence of dynamic scaling in the circuit systems examined, i.e. the impedance (admittance) is invariant with respect to a frequency rescaling $\omega \to \xi\omega$, up to a magnitude scaling factor κ (Alexander and Orbach 1982, Rammal and Toulouse 1983). Its solution takes the form

$$Z(\omega) \propto (\mathrm{i}\omega)^{-\nu} \tag{2}$$

as $\omega \to 0$, with

$$\nu = \ln \kappa / \ln \xi \tag{3}$$

which may be expressed in terms of the admittance, $Y(\omega)$, as

$$Y(\omega) = \{Z(\omega)\}^{-1} \propto (\mathrm{i}\omega)^{\nu}. \tag{4}$$

When $\nu > 0$ the response defined by equations (3) and (4) exhibits a constant phase angle (CPA) and is the typical result obtained for a number of circuit models investigated since Liu (1985) (see for example Kaplan and Gray (1985), Dissado and Hill (1988), Geertsma *et al* (1989), Pajkossy and Nyikos (1989) and Nyikos and Pajkossy (1985)). However, in some specific instances $\nu < 0$ and the results either take the form (Hill and Dissado (1988), Dissado and Hill (1989))

$$Z(\omega) \propto 1 - a(\mathrm{i}\omega)^{|\nu|} \qquad \omega \to 0 \tag{5}$$

or

$$[Y(\omega)/\mathrm{i}\omega] = C(\omega) \propto 1 - b(\mathrm{i}\omega)^{|\nu|} \qquad \omega \to 0 \tag{6}$$

where $C(\omega)$ is the capacitance. It should be noted that equations (5) and (6) apply to *different* model constructions and are not inversely related. In both the above cases the anomalous frequency term appears as a vanishingly small correction to the real component of the complex impedance, equation (5), or capacitance, equation (6), as ω approaches zero. For this reason it may often have been neglected in other investigations. However, it should be noted that the same term dominates the frequency dependence of the imaginary components of $Z(\omega)$ and $C(\omega)$ in the low frequency limits. Since these are directly observable quantities, a proper description of the response does not allow the anomalous frequency term to be regarded only as a correction. Instead equations (5) and (6) define a second class of anomalous response arising from self-similar scaling in the circuit system. This class of behaviour was termed fractional power law response (FPR) in Dissado and Hill (1989).

Although the appearance of anomalous frequency dispersions in the response of self-similar circuit models is interesting in its own right, their importance extends beyond the specific systems for which they were constructed. For example, it has long been recognised that CPA behaviour, equation (2), occurs in the impedance response of a wide range of materials such as biological tissues (Cole 1972, Pethig 1979), dielectrics and systems intermediate between perfect insulators and good conductors (Jonscher 1983). Dielectric (impedance) responses are often described by equivalent circuits (Macdonald 1987) and the recognition that CPA responses could be identified with self-similarity led to the application of this concept to a number of microscopic models (e.g. Klafter and Schlesinger 1986, Köhler and Blumen 1987, Niklasson 1987, Dissado and Hill 1989). The FPR behaviour, equation (6), has also been identified in dielectric responses (Jonscher 1983, Dissado *et al* 1985) and related to self-similarity in a microscopic model (Dissado and Hill 1989).

It is our aim here to aid the development of self-similar models by elucidating the formal rules governing the appearance of each of the two behaviour classes, i.e. CPA and FPR, and particularly the conditions under which FPR will occur. Limitations on the frequency range over which the anomalous frequency dispersions can be observed will also be discussed. Our conclusions will be illustrated by reference to computations of the impedance response of a basic scaled circuit system.

2. Response functions of scaled systems

2.1. *Circuit approach*

The concepts of dynamic scaling in electrical circuits will be illustrated by means of the two examples shown in figure 1. These examples have been chosen because they represent two of the basic possibilities inherent in circuit systems; that in which the

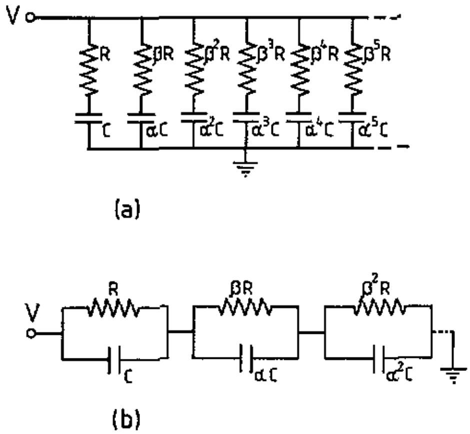

Figure 1. Schematic representation of two scaled electrical circuits. In (a) the sub-circuits are series resistance–capacitance units and the admittance sums as a self-similar series. In (b) the sub-circuits are parallel resistance–capacitance units and the impedance sums as a self-similar series.

voltage drop across scaled sub-units is constant, figure 1(a), and that in which the current through the sub-units is held constant, figure 1(b). Other more complex arrangements can be constructed and these will be discussed elsewhere. In general however, the response of these complex circuits approximates to one or other of the examples chosen over a limited frequency range (e.g. Liu 1985, Dissado and Hill 1988).

In the example of figure 1(a) the scaled sub-units are connected in parallel and this will be termed the parallel mode, PM. The admittance of the system can be written as

$$Y(\omega) = \sum_{u=0}^{N} \frac{\mathrm{i}\omega C\alpha^u}{1 + \mathrm{i}\omega RC\alpha^u\beta^u} \tag{7a}$$

where $\mathrm{i} = (-1)^{1/2}$, α and β are scaling parameters for the elementary capacitance C and resistance R units respectively, and the summation is over the N elements in the circuit. Subsequently throughout this paper we will define the frequency in terms of the characteristic frequency $\omega_0 = (RC)^{-1}$ of the elemental sub-circuit. Using the same notation, the impedance of the series connected circuit (SM) of figure 1(b) can be expressed as

$$Z(\omega/\omega_0) = \sum_{u=0}^{N} \frac{R\beta^u}{1 + \mathrm{i}[\omega/\omega_0]\alpha^u\beta^u} = \sum_{u=0} Z_u(\omega/\omega_0). \tag{8}$$

We note here a fundamental inter-relationship between the two models. By expressing equation (7*a*) in terms of the system capacitance $C(\omega)$ we find

$$\frac{C(\omega/\omega_0)}{C} = R\frac{Y(\omega/\omega_0)}{\mathrm{i}[\omega/\omega_0]} = \sum_{u=0}^{N} \frac{\alpha^u}{1 + \mathrm{i}[\omega/\omega_0]\alpha^u\beta^u}. \tag{7b}$$

It is therefore clear that the behaviour of $C(\omega/\omega_0)/C$ in the parallel mode is the same as that of $Z(\omega/\omega_0)/R$ in the series mode providing the scaling parameters α and β are interchanged. In the rest of this paper we will concentrate our attention on equation (8), since the above inter-relationship allows the complementary behaviour for the parallel model to be deduced immediately.

In spite of the scaled circuit elements it may not be immediately obvious that equation (8) defines a system with dynamic self-similarity. In order to make this explicit we extract the first term of the sum, $Z_0(\omega/\omega_0)$, and renumber the terms in the residual summation. In the limit as N tends to infinity we find

$$Z(\omega/\omega_0) = Z_0(\omega/\omega_0) + \beta Z(\alpha\beta\omega/\omega_0). \tag{9}$$

Thus, provided that

$$Z_0(\omega/\omega_0) \ll \beta Z(\alpha\beta\omega/\omega_0) \tag{10}$$

we recover the dynamic scaling equation (1) with $\beta = \kappa$ and $\alpha\beta = \xi$. This procedure, however, exposes the existence of limits to the frequency range over which the CPA and FPR solutions can be expected to apply.

Consider first the CPA solution

$$Z(\omega/\omega_0) \propto (\mathrm{i}\omega/\omega_0)^{-\nu} \tag{11}$$

with

$$\nu = \ln\beta/(\ln\alpha + \ln\beta) > 0 \tag{12}$$

which requires either $\beta > 1$ and $\alpha\beta > 1$ or $\beta < 1$ and $\alpha\beta < 1$. This behaviour will dominate over both the real and imaginary components of the $Z_0(\omega/\omega_0)$ term

$$Z_0(\omega/\omega_0) = R/(1 + \mathrm{i}\omega/\omega_0) = R[1 - \mathrm{i}\omega/\omega_0]/(1 + [\omega/\omega_0]^2) \tag{13}$$

as ω approaches zero and thus fulfils equation (10) at these frequencies. The CPA behaviour is thus a valid result for low frequencies but only as long as N is infinite. In real systems where N is finite the divergence in equation (11) will be cut off below a limiting frequency. The CPA behaviour, therefore, can be expected to exist only over a limited frequency range, a point which will be taken up later.

Making use of the inter-relationship between the series model and the parallel model allows us to express the PM admittance in the form

$$Y(\omega/\omega_0) \propto (\mathrm{i}\omega/\omega_0)^{\nu}. \tag{14}$$

This is the same as the admittance in the series model, i.e., the inverse of equation (11), and hence there is a complete equivalence of the two models on interchanging α and β for the CPA range of solutions.

If $\beta < 1$ and $\alpha\beta > 1$ or vice versa, the solution to equation (1) takes the form

$$Z(\omega) \propto (\mathrm{i}\omega)^{|\nu|} \tag{15}$$

with

$$|\nu| = \ln\beta/(\ln\alpha + \ln\beta)|. \tag{16}$$

Substitution of equation (15) into equation (9) shows that as $\omega \to 0$ only the imaginary component satisfies the inequality of equation (10). Since the real component of $(\mathrm{i}\omega)^{|\nu|}$ approaches zero as $\omega \to 0$ the real component of the solution to equation (9) will be dominated by that of $Z(\omega/\omega_0)$, i.e. R. Thus $(\mathrm{i}\omega)^{|\nu|}$ cannot, on its own, be a solution to equation (9), nonetheless its contribution cannot be neglected since it dominates the imaginary component at low frequencies. It will be shown later that a solution is allowed only when $\beta < 1$ and $\alpha\beta > 1$ and that its form is the FPR behaviour of equation (5). As for the CPA response, the FPR behaviour satisfies a dynamic scaling equation, only now equation (1) is replaced by the new equation

$$Z(\omega = 0) - Z(\omega) = \kappa[Z(\omega = 0) - Z(\xi\omega)] \qquad \kappa < 1, \xi > 1 \tag{17}$$

which has not been explicitly recognised heretofore. The complementary scaling equation for the parallel model is obtained by replacing the impedance Z in the equation (17) by the capacitance C, leading to a solution of the form of equation (6). With this class of behaviour the admittance of the parallel model does not exhibit the same form of response as that of the series model. Thus unlike the CPA result the two models differ in the form of their response when they obey the FPR scaling equation (17).

2.2. *Fractal properties in scaling*

In this section we show how systems represented by fractal circuits may be reduced to the self-similar ladder circuits (Connor 1972, Clerc *et al* 1990) of figure 1 and relate the scale ratios of the previous section to appropriate fractal dimensions. To this end we consider only the two simple types of circuit that we discussed above, the parallel model and the series model of impedance. Connection to the original circuit can be made via the amplitude scale (α) and the time scale ($\alpha\beta \equiv \xi$) ratios between successive levels p of the embedding. Since α refers to a static property of the system it can be related to the 'size' L_p of the embedding p by a fractal dimension d_f appropriate to the geometry of the system, i.e.

$$\alpha = (L_p/L_{p-1})^{d_f}. \tag{18}$$

In fractal circuit systems ξ is also related to a dimension which we may term d_w, i.e.

$$\xi = (L_p/L_{p-1})^{d_w} \tag{19}$$

and hence the CPA exponent, equation (12), for the impedance response takes the general form given by Liu (1986)

$$\nu = 1 - d_f/d_w. \tag{20}$$

Note that d_w need not be a function of d_f, for example in the electrode surface model proposed by Liu (1985) and Kaplan and Gray (1985) d_w is unity (Liu 1986) and the fractal construction is that of a Cantor bar in one dimension with the other two dimensions remaining Euclidean so that the overall fractal dimension of the system is $d_f + 2$, with $d_f < 1$. On the other hand, the Sierpinski carpet electrode considered by Sapoval (1987) and by Hill and Dissado (1988) has a fractal surface dimension D_p and κ takes the form

$$\kappa = N(L_p/L_{p-1}) = (L_p/L_{p-1})^{1-D_p} \tag{21}$$

where N is the number of new pores of side L_p generated from each pore in the $(p-1)$th embedding with $L_p/L_{p-1} < 1$. In this case

$$\xi = (L_p/L_{p-1})^{-1} \tag{22}$$

and thus when $D_p > 1$, the CPA behaviour is obtained with $d_f = -(D_p - 1)$ and $d_w = -1$, i.e.

$$\nu = 2 - D_p = 3 - D_v \tag{23}$$

where D_v is the volume fractal dimension. When, however, $D_p < 1$, then $\kappa < 1$, $\xi > 1$ and the FPR behaviour is found with

$$|\nu| = 1 - D_p \tag{24}$$

from equation (16).

The two examples considered show how simple fractal circuits can be reduced to one of the two general forms examined here and in what sense the logarithms of the scale ratios can be considered as related to the fractal dimension of the system geometry and timescale. More complex fractal circuits than the examples given here also reduce to one or other of the two forms considered, at least over a finite range of embeddings. It should be noted that in these cases d_w may take other values than that of unity as found in the two examples quoted, e.g. for a D-dimensional Sierpínski gasket (Liu 1986) d_w is given by $\ln(D+3)/\ln 2$. In order to retain generality in our discussion, and not be tied to a specific fractal construction, we have developed our arguments in the scaled circuit formalism for arbitrary scale ratios α and β in capacitance and resistance respectively.

2.3. *Some formal solutions*

In section 2.1 we considered our system to be composed of a scaled sum of elementary circuit elements which lead to a governing equation, equation (8), of the form

$$S_N(x) = \sum_{u=0}^{N} \kappa^u f(\xi^u x) \tag{25}$$

where $x = \mathrm{i}\omega/\omega_0$. In the basic circuit construction of section 2.1 $f(x)$ has the form $f(x) = (1+x)^{-1}$, however the sub-units may in general be much more complex that those of figure 1 with different functional forms for $f(x)$. Here we consider some forms for $f(x)$ which are analytically solvable in the limit of $N \to \infty$.

2.3.1. Polynomial functions. Many functions can be expressed as infinite order polynomial series, i.e.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{26}$$

Expanding the series, applying the summation of equation (25) to each term and collecting terms of the same power in x gives

$$S(x) = a_0(1/(1-\kappa)) + a_1(x/(1-\kappa\xi)) + a_2(x^2/(1-\kappa\xi^2)) + \ldots = \sum_{k=0}^{\infty} c_k x^k \tag{27}$$

with

$$c_k = a_k(1-\kappa\xi^k)^{-1}. \tag{28}$$

It should be noted that for these cases $S(x)$ is finite in the $N \to \infty$ limit when $\kappa < 1$ and $\xi < 1$.

2.3.2. Fractional powers. By definition, self-similar systems retain the same scaling when they are constructed from scale-related sub-groups as they do in the terms of the elementary units. In this case the appropriate expression for $f(x)$ of the sub-group will be either the CPA behaviour of equation (2) or the FPR behaviour of equation (5). Furthermore, the functional form of $S(x)$ must be the same as that of $f(x)$ if the system is dynamically self-similar. This invariance can be demonstrated by substituting

$$f(x) = Ax^{-\alpha} \tag{29}$$

into equation (25) which gives

$$S(x) = Ax^{-\alpha}(1 + \kappa\xi^{-\alpha} + \kappa^2\xi^{-2\alpha} + \ldots). \tag{30}$$

Table 1. Examples of solutions to equation (25) for N approaching infinity. Note: (i) From the first line in the table we see that any function that can be expanded as a power series will give a solution. Other examples are given in the following three lines. As we have already noted, any linear combination of solutions is also a solution. (ii) Functions with a non-integer power exponent are the only functions which are invariant in form under summation. These are given in the last two lines of the table.

$f(x)$	$S(x)$	Conditions
$\sum_{k=0}^{\infty} a_k x^k$	$\sum_{k=0}^{\infty} a_k \frac{x^k}{1-\kappa\xi^k}$	$\xi<1; \kappa<1$
$\frac{A}{1+x}$	$A\sum_{k=0}^{\infty} \frac{\kappa^k}{1+x\xi^k}$	$\xi<1; \kappa<1$ and $\xi>\kappa>1$
$A\,e^{-x}$	$A\sum_{k=0}^{\infty} [-1]^k \frac{x^k}{k![1-k\xi^k]}$	$\xi<1; \kappa<1$
$A\exp(-ax^p)$	$A\sum_{k=0}^{\infty} [-1]^k \frac{a^k x^{kp}}{k![1-\kappa\xi^{kp}]}$	$\xi<1; \kappa<1$
$Ax^{-\alpha}$	$A\frac{x^{-\alpha}}{1-\kappa\xi^{-\alpha}}$	$\alpha>\frac{\ln\kappa}{\ln\xi}$
$A[1-ax^{\alpha}]$	$\frac{A}{1-\kappa}\left[1-\frac{a[1-\kappa]}{1-\kappa\xi^{\alpha}}x^{\alpha}\right]$	$\alpha<\frac{\ln[1/\kappa]}{\ln\xi}; \kappa<1$

$S(x)$ therefore retains the functional form of $f(x)$ and it is interesting to note that this remains the case even when self-similarity within the sub-groups is different from that of the sub-group system, i.e. $\alpha \neq \ln\kappa/\ln\xi$, with $S(x)$ remaining finite, at finite frequency in the limit $N\to\infty$ as long as $\kappa\xi^{-\alpha}<1$, i.e.

$$\alpha > \ln\kappa/\ln\xi. \tag{31}$$

When the system retains its self-similarity throughout, i.e. $\alpha = \ln\kappa/\ln\xi$ then $S(x)$ is proportional to N (the number of sub-groups)., This has the effect of renormalising the frequency scale ω_0 to $\omega_0 N^{1/\alpha}$ and hence of altering the upper frequency limit to the CPA behaviour in finite systems. An FPR function, equation (5), for $f(x)$ can be dealt with in a similar manner to the CPA behaviour of equation (30) and shown to be invariant in form under the self-similar summation of equation (25). Again, the frequency scale is renormalised when the self-similarity remains unchanged from that of the sub-group but this time with ω_0 converted to $\omega_0/N^{1/\alpha}$.

A number of functions which gives analytical solutions for $S(x)$ in equation (25) according to sections 2.3.1 and 2.3.2 are listed in table 1. It should be noted that any arbitrary linear combination of solutions will also be a solution for $S(x)$, and thus in general the range of possible solutions is large. Only the CPA and FPR behaviours however are invariant to self-similar rescaling and are therefore the allowed solutions when dynamic scaling exists in the system. Other patterns of behaviour represented by different $f(x)$ therefore express the extent to which the system deviates from dynamic self-similarity. However the CPA and FPR forms must themselves be generated by a scaled system whose elementary circuit has a simple form for $f(x)$ such as that described in section 2.1. Therefore some, if not all, of the algebraic expressions for $f(x)$ listed in table

1 must be equivalent either to the CPA or FPR behaviour over a finite range of x. This behaviour is hidden in the infinite summation representation and so we now turn to a formalism that explicitly reveals the dynamic scaling.

2.4. *Integral formalism*

In section 2.1 we showed how two scaled circuit models with

$$f(x) = A(1+x)^{-1} \tag{32}$$

could be solved to yield either CPA or FPR behaviour for the response over a system dependent frequency range. The exact solution obtained in section 2.3 however gives $S(x)$ in the form of a polynomial series. Reconciliation of the two approaches requires that the polynomial series approximates to the power law dependence in the CPA or FPR behaviour for a limited range of x. It is therefore possible that other functions for $f(x)$ may behave in a similar manner even although this is not obvious from table 1. This difficulty is removed by adopting an integral approach to equation (25), which we now write as

$$S_N(x) = \int_0^N \kappa^u f[\xi^u x]\, \mathrm{d}u. \tag{33}$$

This formalism not only allows us to deal with all integrable functions $f(x)$ but also encompasses the possibility of self-similarity for which the scaling is continuous rather than by discrete steps (i.e. stochastic as well as deterministic fractals).

Letting $\xi^u x = z$ be the new variable gives

$$S_N(x) = (x^{-\nu}/\ln\xi) \int_x^{x\xi^N} z^{\nu-1} f(z)\, \mathrm{d}z \tag{34}$$

where $\nu = \ln\kappa/\ln\xi$ is the dynamic scaling exponent, equation (3). In this form we have abstracted the fractional power law dependence from the integral and the problem becomes one of defining the conditions under which the integral exists and is independent of x. Here the behaviour of $f(z)$ as the limits $z \to 0$ and $z \to \infty$ are approached is of central importance.

We now discuss the possible solutions for $S_N(x)$.

(a) $\xi > 1, \kappa > 1$ (i.e. $\nu > 0$): a solution for $S_N(x)$ can now be sought in the limit $x \to 0$, but with N large enough so that at finite x, $\xi^N x$ approaches infinity. Equation (34) now becomes

$$S_N(x) \simeq (x^{-\nu}/\ln\xi) \int_0^\infty z^{\nu-1} f(z)\, \mathrm{d}z \simeq (x^{-\nu}/\ln\xi) f^*(\nu) \qquad x \to 0 \tag{35}$$

where $f^*(\nu)$ is the Mellin transform of $f(z)$ (Sneddon 1972). This solution exists when

$f^*(z)$ is finite, and a cursory examination of the integral shows that this is ensured for $\nu < 1$ by the following restriction on the limiting behaviour of $f(z)$, i.e.

$$f(z) \to \begin{cases} \text{constant or zero as } z \to 0 \\ z^{-p}\ (p = 1, 2, 3, \ldots) \qquad z \to \infty. \end{cases} \tag{36}$$

(b) $\xi < 1$, $\kappa < 1$ (i.e. $\nu > 0$): here we take $x \to \infty$ but $\xi^N x$ as small and approaching zero. In this case we find

$$S_N(x) \simeq (x^{-\nu}/\ln \xi) \int_\infty^0 z^{\nu-1} f(z)\,\mathrm{d}z = (x^{-\nu}/\ln[1/\xi]) \int_0^\infty z^{\nu-1} f(z)\,\mathrm{d}z$$

$$\simeq (x^{-\nu}/\ln 1/\xi) f^*(\nu) \qquad x \to \infty \tag{37}$$

which is finite under the same conditions, equations (36), as the solution in (a), i.e. equation (35).

These two results both represent a CPA solution. The differences in valid frequency ranges ($x \to 0$, $x \to \infty$) refer only to the fact that in (a) the CPA behaviour appears at frequencies below ω_0, whereas in (b) it appears above ω_0. It should be noted that the circuit model function $f(x) = (1 + x)^{-1}$ complies with the restrictions (36) and hence gives the CPA behaviour under these conditions as deduced in section 2.1. The series model to parallel model inter-relationship of section 2.1 gives the complementary admittance expressions. However equation (7*a*) can be used to set up an equivalent expression to equation (34) with $\nu < 0$ instead of $\nu > 0$. In this case

$$Y(x) \equiv S_N(x) \propto x^{|\nu|} \qquad \xi > 1, \kappa < 1 \text{ or } \xi < 1, \kappa > 1 \tag{38}$$

if

$$f(z) \to \begin{cases} \text{constant} & z \to \infty \\ z^p & (p = 1, 2, 3, \ldots), z \to 0. \end{cases} \tag{39}$$

Examination of equation (7*a*) shows that the appropriate $f(x) = x/(1 + x)$ meets the requirements of equation (39), and hence the CPA results of section 2.1 are consistent with the solutions obtained in this section for more general forms of $f(x)$.

(c) When $\nu < 0$ and

$$f(z) \to \text{constant} \qquad z \to 0 \tag{40a}$$

a solution of the form of equation (35) or equation (37) is no longer valid because $f^*(\nu < 0)$ is no longer finite. In this case we have to integrate equation (34) by parts which gives

$$S_N(x) = (x^{|\nu|}/|\nu| \ln \xi) \left\{ [-z^{|\nu|} f(z)]_x^{x\xi^N} + \int_x^{x\xi^N} z^{-|\nu|} f'(z)\,\mathrm{d}z \right\} \tag{41}$$

$$= (1/|\nu| \ln \xi) \left\{ f(x) - \xi^{-N|\nu|} f(\xi^N x) + x^{|\nu|} \int_x^{x\xi^N} z^{-|\nu|} f'(z)\,\mathrm{d}z \right\} \tag{42}$$

where $f'(z) = \mathrm{d}f(z)/\mathrm{d}z$.

The existence of a solution now depends on the behaviour of $f'(z)$ and $f(z)$ as the limits of zero and infinity for z are approached. For the case where $\xi > 1$ ($\kappa < 1$) we look for a solution as $x \to 0$ and $\xi^N x \to \infty$. Here we require that

$$|f(\xi^N x)/\xi^{N|\nu|}| \ll 1 \qquad \text{as } \xi^N, \xi^N x, \to \infty$$

i.e.

$$f(z) \to \text{constant or zero as } z \to \infty. \tag{40b}$$

The integral in equation (42) must also be finite and for this to be the case

$$f'(z) \to \begin{cases} \text{constant or zero} & z \to 0 \\ z^{-p} & (p = 1, 2, 3) \quad z \to \infty \end{cases}. \tag{43}$$

When these conditions are met the solution takes the form

$$S_N(x) = (1/|\nu| \ln \xi)\left[f(0) + x^{|\nu|}\int_0^\infty z^{-|\nu|}f'(z)\,\mathrm{d}z\right] \qquad x \to 0. \tag{44}$$

We note that $f(z) = (1+z)^{-1}$, appropriate to the circuit models of section 2.1, fulfils the above conditions on $f(z)$ and $f'(z) = -(1+z)^{-2}$. In this case $S_N(x)$ takes the form of the FPR behaviour, equation (5). To make this more explicit we rewrite $S_N(x)$ as

$$S_N(x) = (1/|\nu| \ln \xi)\left[f(0) - x^{|\nu|}\int_0^\infty z^{-|\nu|}(-f'(z))\,\mathrm{d}z\right]. \tag{45}$$

The restrictions given in equations (40*a*), (40*b*) and (43) on $f(z)$ and $f'(z)$ therefore define the conditions under which the FPR form of response occurs for $\xi > 1$, $\kappa < 1$. In the alternative range of ξ for which $\nu < 0$ (i.e. $\xi < 1$, $\kappa > 1$) and for which we have to seek a solution in the $x \to \infty$ range, there is no solution for $S_N(x)$. Thus, unlike the CPA, the FPR behaviour does not exist as a high frequency limit with respect to the frequency of the elementary circuit, ω_0. The FPR is thus inherently a low ($\omega \to 0$) frequency response, which the inter-relationship of series and parallel circuit models shows may be observed either as an impedance or capacitance response when the required conditions are satisfied.

2.5. *Ranges of validity*

In the previous section we derived expressions for $S_N(x)$ by assuming that the lower and upper limits in the integrals of equations (34) and (41) could be extended to zero and infinity. For systems of a finite size this approximation is only valid for a specific frequency range which can be obtained by determining the conditions for which the residual terms are negligible. As the techniques are quite general we shall consider only one case, case (b), in detail and list the equivalent conditions for cases (a) and (c) in table 2.

One of the conditions assumed in replacing equation (34) by equation (37) is that

$$(x^{-\nu}/\ln[1/\xi])\int_0^{x\xi^N} z^{\nu-1}f(z)\,\mathrm{d}z \tag{46}$$

approaches zero when $\xi < 1$ if x is large but $\xi^N x \to 0$. As z is small over the whole

Table 2. Ranges of validity for the integral formalism of section 2.4. See section 2.5 for details. $f^*(\nu)$ is the Mellin transform $\int_0^\infty z^{\nu-1} f(z)\,dz$; A_n and B_n are the nth coefficients of the expansion of $f(z)$ in the series in z and z^{-1} respectively; A_0' is the equivalent coefficient in the expansion of the derivative $f'(z)$.

	Range of					
Case	ν	ξ	κ	$x \gg$	$x \ll$	$S_N(x)$
(a)	$0<\nu<1$	>1	>1	$\dfrac{B_1}{\ln\xi[1-\nu]\xi^{N[1-\nu]}}$	$\dfrac{\ln\xi[1+\nu]}{A_1}$	$\dfrac{x^{-\nu}}{\ln\xi} f^*(\nu)$
	$-1<\nu<0$		<1			
(b)	$0<\nu<1$	<1	<1	$\dfrac{B_1}{\ln[1/\xi][1-\nu]}$	$\dfrac{\ln[1/\xi][1+\nu]}{A_1\xi^{N[1+\nu]}}$	$\dfrac{x^{-\nu}}{\ln[1/\xi]} f^*(\nu)$
	$-1<\nu<0$		>1			
(c)	$-1<\nu<0$	>1	<1	Indeterminate	$\dfrac{\ln\xi[1-\|\nu\|]\|\nu\|}{A_0'}$	$\dfrac{1}{\|\nu\|\ln\xi}\left[f(0) - x^{\|\nu\|}\int_0^\infty z^{-\|\nu\|}\{-f'(z)\}\,dz\right]$

frequency range, and $f(z)$ must approach either zero or a constant as $z \to 0$, see equation (36), we can represent $f(z)$ by a power series $\Sigma_{k=0} A_k z^k$ to get

$$(x^{-\nu}/\ln[1/\xi]) \int_0^{x\xi^N} \sum_{k=0} A_k z^{k+\nu+1}\,dz$$
$$= (x^{-\nu}/\ln[1/\xi]) \sum_{k=0} A_k[\xi^N x]^{k+\nu}(k+\nu)^{-1} \to 0. \qquad (47)$$

Only the first two terms in the series are important and these give

$$(A_0\xi^N/\nu \ln[1/\xi]) + (A_1\xi^{N[\nu+1]}x/[1+\nu]\ln[1/\xi]) \ll 1 \qquad (48)$$

which, in the range $\xi < 1$ and large N (i.e. $\xi^{-N\nu} \gg 1$), reduces to

$$x \ll \frac{1}{A_1}\ln(1/\xi)(1+\nu)\xi^{-N(\nu+1)} \qquad (49)$$

as an upper bound to the frequency range of validity of equation (37).

The second condition comes from the requirement that

$$(x^{-\nu}/\ln[1/\xi]) \int_x^\infty z^{\nu-1} f(z)\,dz \to 0 \qquad x \to \infty. \qquad (50)$$

Here we can make use of the restriction, equation (36), to write $f(z)$ as $\Sigma_{k=1}^{\infty} B_k z^{-k}$. Substituting for $f(z)$ and integrating gives

$$(x^{-\nu}/\ln[1/\xi])\left[\sum_{k=1}^{\infty} B_k \frac{z^{\nu-k}}{\nu-k}\right]_x^{\infty} \to 0 \qquad x \to \infty \qquad (51)$$

and in this case we find a lower bound in the range of x for which the CPA solution, equation (37), is valid, i.e.

$$x \gg B_1[\ln(1/\xi)(1-\nu)]^{-1}. \qquad (52)$$

Equivalent bounds on the frequency ranges of the other two cases have been derived and are listed in table 2.

3. Calculated response functions

The polynomial approach developed in section 2.3 is an ideal basis for the computation of responses to any order in N for a suitable sub-group function $f(x)$. The function $f(x)$ is itself a spectral function but need not be either a relaxation function or a damped resonance. There are, however, limitations on the type of function that can be used in that it must yield a summable series on rescaling. Two such functions have been chosen; the relaxation functions of the simple circuit models of section 2.1, and an exponential frequency response. We report the dispersions calculated for these functions and examine the frequency range of validity obtained from section 2.5 which are summarised in table 2.

3.1. The function $(1+x)^{-1}$

In addition to being the response in the elementary circuit in the models of figure 1, $(1+x)^{-1}$ is the response function of a dipole relaxing in a viscous medium independently

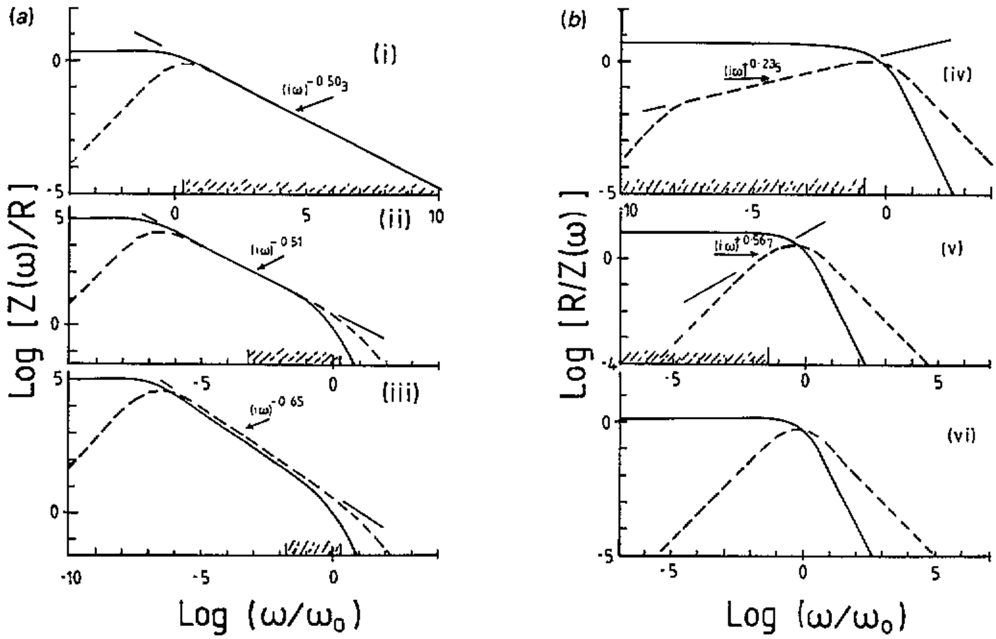

Figure 2. Log–log plots of the frequency dependence of the SM-impedance, equation (13) [or, equivalently, the PM-capacitance of equation (7*b*)] with $f(x) = (1 + x)^{-1}$. The values of the scaling parameters are listed in table 3 together with the calculated and observed values of the exponent ν. The shaded regions on the frequency axis indicate the regions of validity of the scaling behaviour as determined in section 2.5. The real components of the responses are shown by the full curves and the imaginary components by the broken curves. CPA behaviour is shown in (i) to (iii) and FPR behaviour in (iv) and (v).

of other dipoles and without modifying its environment (Debye 1945), normalized to unit magnitude and loss peak frequency, i.e.

$$f(x) = 1/(1 + x) = 1/[1 + (\omega/\omega_0)^2] - \mathrm{i}(\omega/\omega_0)/[1 + (\omega/\omega_0)^2]. \qquad (53)$$

For this form of $f(x)$ the computed responses show that the Debye behaviour of equation (53) is modified when the dipoles form a self-similar system scaled both in frequency and magnitude. In figure 2(*a*) and (*b*) we present, diagrammatically, the computed responses in log/log plots for a set of values of the scaling parameters κ and ξ. A summary of the information contained in these plots is given in table 3. In the table we also list the frequency ranges over which the CPA or FPR approximations are calculated to apply from section 2.5, and the allowable ranges are indicated in the figures. It is seen that the calculated and computed exponents are satisfactorily close and that, to a degree, there is agreement between the postulated and computed ranges over which the anomalous frequency behaviour is found. Examples of all three parameter ranges, i.e. cases (a), (b) and (c) of section 2.4, are reported together with a single case for which the parameters violate the conditions of dynamic scaling. In the latter case the response reverts to the form given in equation (53). It should be noted that outside the frequency range where CPA or FPR behaviour develops the computed response assumes the form of equation (53), i.e. that the limiting terms in the finite N summation form for $S(x)$, equation (25), dominate in these regions.

Table 3. Parameters used to obtain the responses presented in figures 2 and 3.

Figure and plot No.	ξ	κ	ν	Case	N	$x \gg$	$x \ll$	$\nu_{observed}$
2[a] i	0.25	0.5	0.5	(b)	20	1.5	2.4×10^{18}	0.503
ii	2.25	1.5	0.5	(a)	20	7.4×10^{-4}	1.2	0.51
iii	2.21	1.7	0.667	(a)	20	1.9×10^{-2}	1.3	0.65
iv	2.5	0.8	−0.243	(c)	20	0	0.17	−0.235
v	1.2	0.9	−0.578	(c)	20	0	4.4×10^{-2}	−0.567
vi	2.5	0.3	−1.31	—	20	—	—	Out of range
3[b] i	1.1	0.95	−0.538	(c)	100	0	2.4×10^{-2}	−0.46
ii	1.05	0.97	−0.624	(c)	300	0	1.1×10^{-2}	−0.58
iii	1.03	0.985	−0.511	(c)	500	0	7.4×10^{-3}	−0.50

[a] For $f(x) = (1 + x)^{-1}$; $A_0 = 1$; $A_1 = -1$; $A'_0 = 1$; $A'_1 = 2$; $B_1 = 1$.
[b] For $f(x) = \exp(-x)$, $A_1 = -1$; $A'_0 = 1$; $A'_1 = 1$. A_n, A'_n and B_n are as defined in table 2.

3.2. *The function* exp(−x)

As an example of a function which is not a dispersion relationship we have chosen

$$f(x) = \exp(-x) = \cos(\omega/\omega_0) - \mathrm{i}\sin(\omega/\omega_0). \tag{54}$$

In this case $S(x)$ could only be calculated with any accuracy for a limited range of κ and ξ close to unity and this required large values of N. For x greater than unity the oscillatory nature of $f(x)$ in equation (46) led to considerable difficulty in obtaining convergent solutions and hence this range of frequencies is not reported. However even for this highly artificial elementary response function the FPR is able to dominate the response over a given frequency range, as can be seen in figure 3. It therefore seems likely that an anomalous frequency dependnece of the CPA or FPR type will develop over a limited frequency range in scaled systems regardless of the functional form of the response of the combined elements which make up the scaled elementary unit.

4. Discussion and summary

In this paper we have identified the existence of two distinct classes of dynamic scaling equations, equations (1) and (17), for the response of scaled circuit systems. Such systems are important for the understanding of dielectric responses originating in either molecular dipole or ion displacements because the relaxation response of an independent site dipole can be represented by the series resistance–capacitance of the elementary circuit. This equivalence is the basis for the equivalent circuit description of the dielectric response of materials (Macdonald 1987). Formally the capacitance response $f(x)$, of a molecular (site) dipole is given by the Laplace transform (imaginary argument) of a response function (Kubo 1957) which must be constant at zero time and approach zero as time approaches infinity, hence $f(x)$ must tend to zero as $x \to \infty$ and to a constant as $x \to 0$. Therefore in the dielectric relaxation (i.e. capacitance response) of molecular dipolar systems as of that of true local circuits, $f(x)$ has a form which meets the requirements (see section 2.4) for the appearance of CPA or FPR behaviour, provided that the dipole magnitudes and relaxation times scale appropriately.

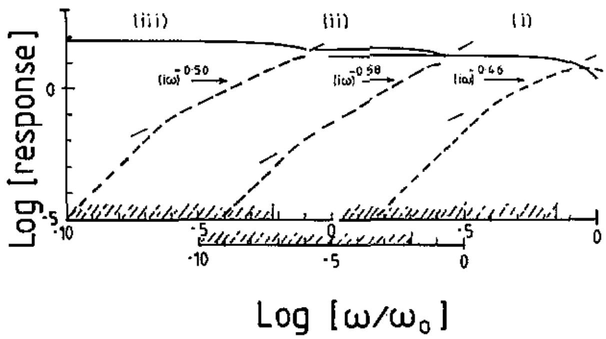

Figure 3. An equivalent set of plots to those presented in figure 2 but using the function $f(x) = \exp(-x)$. The relevant parameters for these plots are also presented in table 3.

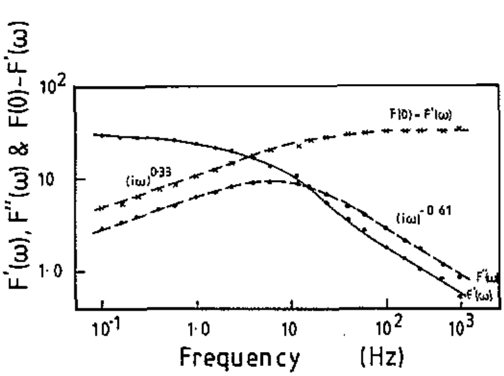

Figure 4. Log–log plots of the dielectric susceptibility [∝ capacitance] of a crystal of TGS as measured and reported by Pawlaczyk *et al* (1978). The real, $F'(\omega)$, and imaginary $F''(\omega)$, component values have been plotted as circles and the values for $F'(\omega = 0) - F'(\omega)$ as crosses. The latter obeys the FPR dynamic scaling equation (17) at frequencies below 3 Hz and is thus equivalent in form to case (c) of table 3, see figure 3.

The equivalence between $f(x)$ for a relaxating molecular dipole and an elementary circuit is a purely formal one and does not imply the existence of actual resistance and capacitance elements such as might be expected in biological tissues, for example (Dissado 1990), however it does allow the capacitance and impedance responses of both types of material system to be considered in the same theoretical framework. Here we have concentrated on establishing the rules which govern the appearance of the two different classes of dynamic scaling. These rules are expressed in terms of the amplitude (κ) and timescale (ξ) ratios in a scaled circuit system, which is an essential requirement for the appearance of either of the two dynamic scaling behaviours, i.e. the CPA and FPR classes. We have adopted this approach in order to retain generality in our results. Specific fractal model constructions will reduce to one or other of our scaling circuit systems with specific values for κ and ξ dependent upon the appropriate fractal dimensions, as we have described in section 2.2. The rules can thus be used quite generally to determine the class of dynamic response for a given fractal system and the value of the anomalous exponent to be expected. Parameter ranges for which the response will show no anaomalous behaviour, regardless of the existence of scaling, have also been determined.

It should be noted that physical systems have a finite size and hence any fractal structure they may possess will be limited to a finite number of embeddings (N). In this case the anomalous CPA or FPR response will only appear over a limited frequency range. Analytical expressions for the frequencies at which the anomalous responses are truncated have been obtained in section 2.4 for the case in which the system response can be represented by an integral formalism. Such a formalism is likely to be more appropriate to the stochastic fractals of physical systems than to the constructed deterministic circuits described in section 3. In this latter case the results show that anomalous behaviour may be found for relatively small numbers of embeddings ($N = 20$) (see table 3) but that their range of validity as determined in section 2.4 only gave a guide to the computed range. Better agreement can be expected if large values of N are used or when the fractal circuit construction is stochastic.

Of the two classes of dynamic scaling identified here, the first class is well known (see Liu 1985) and obeys the equation

$$Z(\omega) = \kappa Z(\xi\omega) \qquad \kappa, \xi > 1 \text{ or } \kappa, \xi < 1 \tag{1}$$

which has a solution in the form of the CPA behaviour

$$Z(\omega) \propto (\mathrm{i}\omega)^{-\nu} \qquad 1 > \nu = \ln \kappa/\ln \xi = 1 - d_{\mathrm{f}}/d_{\mathrm{w}} > 0 \tag{2}$$

and has long been recognised to exist in the dielectric response of materials (see Cole 1972, Jonscher 1983). An origin in terms of dynamically scaled dipole relaxations has been incorporated into many molecular models (e.g. Klafter and Schlesinger 1986, Dissado and Hill 1989). Intuitively it is easy to see how such a scaling system may be established once it is acknowledged that the motion of the individual dipoles can couple to displacements in their environment and hence indirectly to one another. Just as with systems approaching a critical point (Hohenberg and Halperin 1977), the larger the total dipole of the group ($\kappa > 1$) the slower the response ($\xi > 1$) and the CPA response follows if the groups are self-similarly related (e.g. Dissado and Hill 1987).

The second class of dynamic scaling equation we have derived, i.e.

$$F(\omega = 0) - F(\omega) = \kappa[F(\omega = 0) - F(\xi\omega)] \qquad \kappa < 1, \xi < 1 \tag{17}$$

with

$$F(\omega) = Z(\omega) \text{ or } C(\omega)$$

has not previously been given explicit form though its existence has been foreshadowed on the basis of the analysis of experimental data (Dissado and Hill 1987). The solution to this class of dynamic scaling equation is the FPR behaviour

$$F(\omega) \propto 1 - a(\mathrm{i}\omega)^{|\nu|} \qquad 1 > |\nu| = \ln[1/\kappa]/\ln \xi > 0 \tag{5}$$

which has also been recognised in the experimental response of many materials (see Jonscher 1983, Hill 1978), an example of which is given in figure 4. Specific circuit models have been shown to give this reponse (Hill and Dissado 1988, Dissado and Hill 1988) although only one molecular model, to date, (Dissado and Hill 1989) incorporates the FPR in its response function. One reason for this may be the difficulty in recognising its appearance, although the application of equation (17), as shown in figure 4, should make this easier. Another reason is likely to be the difficulty associated with envisaging a molecular situation in which the dielectric increment contributed by a sub-group reduces ($\kappa < 1$) as the relaxation rate reduces ($\xi > 1$), i.e. the smaller the group dipole the slower its relaxation. However, one type of system comes to mind. Consider a system in which small regions are constrained from moving by larger regions that surround them and for which they, in their turn, constrain smaller units, etc. Such a picture may well apply to a glassy material. In this case the smaller groups cannot relax until the larger groups have first responded and a dynamic scaling relationship between size and relaxation time will lead to an FPR behaviour at low frequencies. Although this picture is similar to that proposed by Palmer *et al* (1984) for the α-process in glass forming systems, its identification with the new scaling class leads to an exact behavioural form (equation (5)) which differs from the approximate form obtained by these workers.

In their simplest form the requirements for FPR behaviour define a system that is approaching equilibrium, namely that the incremental contribution of relaxing groups gets progressively smaller as equilibrium is approached, with the groups being self-similarly related. It has been argued, on the basis of experimental observation, that this is a general feature of the linear regression (Dissado and Hill 1987) of structural fluctuations. The identification here of the dynamic scaling equation and the derivation of the conditions governing its appearance can therefore be expected to contribute to an understanding of the approach to equilibrium of natural fluctuations in real systems.

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